noting that, for $m=2$, the double sum is not present in the above expression.

Using the results from Eqs. (3) and (6) in conjunction with the aforementioned constraints ($\Delta V_T$, longitude change, maximum $\Delta V$ per burn, minimum number of maneuvers) allows a unique solution to the station acquisition problem to be found. The user of this technique must use some care as the first time step $\Delta t_1$ must be specified and the appropriate apsis location should be ensured for all maneuvers. Nonetheless, this approach allows for an analytical solution to a problem that nominally requires a large software system.

## Example Maneuver Design

As an example consider the situation summarized in Table 1. Using the minimum coast-time relation [Eq. (3)], the value for drift acceleration $\ddot{\lambda}$ and acceleration variation $\partial\ddot{\lambda}/\partial\lambda$ (which can be calculated from equations in Ref. 7), in addition to the other available information yields the time to acquire station—10.86 days. Consequently, the equation describing the continuous burn strategy is

$$f(t)=0.357t^2-7.748t \text{ deg}$$

where time $t$ is measured in days.

The procedure used in calculating the various coast time intervals [Eq. (6)] can now be used. Table 2 shows the values calculated as a preliminary estimate of the maneuver sequence. Note that the initial selected coast time has been set to 1 day and that the values have been rounded off or modified to put the satellite at the correct point in the orbit. While the amount of time to reach the station nearly matches the ideal, the longitude change is far short of that needed. However, it is easily observed that the difference is nearly equal (within 0.6%) to an additional 2 days of coast at the first maneuver drift rate. Thus, with this additional time, a unique solution has been found, using minimum velocity change and minimum number of maneuvers to enable the acquisition of the desired station in a nearly optimal fashion. Figure 1 shows the comparison between the initial and final designs and the continuous burn optimal path.

## Conclusion

The analytical technique developed in this Note permits the estimation of an optimal maneuver sequence without enlisting the aid of massive computer software systems. It does this by minimizing the deviation between a finite approximation and the exact optimal solution to the problem of station acquisition. The solution found uses the minimum velocity change and minimum number of maneuvers to make a fixed longitude change. The approach taken is simple and powerful, allowing the analyst considerable insight and control over the maneuver sequences. An example demonstrates the utility of the approach.

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# Equations of Attitude Motion for an *N*-Body Satellite with Moving Joints

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## Nomenclature

$\underset{\sim}{F}_\lambda$ –total external force on body $\lambda$
$\underset{\sim}{F}_\lambda$ = nongravitational external force on body $\lambda$
$\underset{\sim}{F}^H_{\lambda j}$ = interaction force on body $\lambda$ transmitted through joint $j$
$\hat{\underset{\sim}{g}}_j$ = unit vector along rotation axis of joint
$J_\lambda$ = the set of joints on body $\lambda$
$m$ = total mass
$m_\lambda$ = mass of body $\lambda$
$r$ = number of rotational degrees of freedom
$S$ = the set of bodies in the topological tree
$\underset{\sim}{T}_\lambda$ = total external torque on body $\lambda$
$\underset{\sim}{T}'_\lambda$ = nongravitational external torque on body $\lambda$
$\underset{\sim}{T}^C_{\lambda j}$ = gimbal constraint torque on body $\lambda$ at joint $j$
$\underset{\sim}{T}^H_{\lambda j}$ = torque on body $\lambda$ transmitted through joint $j$
$\underset{\sim}{T}^{S,D}_{\lambda j}$ = spring-damper torque on body $\lambda$ at joint $j$
$\gamma$ = planet's gravitational constant
$\gamma_i$ = angle of rotation about axis $\hat{\underset{\sim}{g}}_i$
$\underset{\sim}{\rho}$ = planetocentric position vector of satellite composite c.m.
$\hat{\underset{\sim}{\rho}}$ = unit vector in direction of $\rho$
$\hat{\rho}_\lambda$ = planetocentric position vector of c.m. of body $\lambda$
$\Phi_\lambda$ = inertia dyadic of body $\lambda$ about center of mass
$\underset{\sim}{\omega}_0$ –angular velocity of the reference body
$\underset{\sim}{\omega}_\lambda$ = angular velocity of body $\lambda$
$\underset{\sim}{1}$ = unit dyadic

## Introduction

THE equations of spacecraft attitude dynamics derived by Hooker and Margulies[1] and their subsequent modification by Hooker[2] are well known and widely used.[3,4] This formulation is appropriate for the motion of a system of $N$ rigid bodies connected by dissipative elastic joints and subject to arbitrary external forces and torques. The derivation follows an Eulerian method which accounts for internal and external torques in a straightforward way. The axes of rotation of each joint in the system and internal torque laws about these axes may be prescribed. Constraint torques at the joints do not appear in the final equations of motion, but their magnitudes may be determined a posteriori.[2] Two restrictions are imposed on the system in this formulation; chains of connected bodies may not form closed loops, and joint positions must be fixed with respect to the bodies connected at the joint.

Received Dec. 9, 1983; revision received Sept. 18, 1984. 

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In recent research concerned with the attitude motion of a robot spacecraft, sometimes called a teleoperator, we have needed to determine the attitude equations of motion for a system of connected rigid bodies when a joint is allowed to move along the surface of one of the bodies. Such a case is not considered in the Hooker-Margulies equations. We have modified these equations to allow prescribed motions of any of the joints in the system with respect to the bodies which they connect. In the remainder of this Note we will present these additions to the *H-M* equations, in the terminology of the original formulation, together with sufficient results and definitions from the original formulation[2] to enable a reader unfamiliar with the *H-M* equations to employ our main result.

## Analysis

Following the development and notation of Ref. 1, Newton's and Euler's equations for body $\lambda$ are:

$$F_\lambda + \sum_{j\epsilon J_\lambda} F^H_{\lambda j} = m_\lambda \ddot{\rho}_\lambda \tag{1}$$

$$\Phi_\lambda \cdot \dot{\omega}_\lambda + \omega_\lambda \times \Phi_\lambda \cdot \omega_\lambda = T_\lambda + \sum_{j\epsilon J_\lambda} T^H_{\lambda j} + \sum_{j\epsilon J_\lambda} \mathcal{L}_{\lambda j} \times F^H_{\lambda j} \tag{2}$$

where $\mathcal{L}_{\lambda j}$ is the vector from the center of mass of body $\lambda$ to the joint $j$ on body $\lambda$. The hinge forces $F^H_{\lambda j}$ are then eliminated from Euler's equation by using Newton's equation and summing it over the bodies $\mu$ connected to body $\lambda$ at joint $j$. This yields

$$\sum_{j\epsilon J_\lambda} \mathcal{L}_{\lambda j} \times F^H_{\lambda j} = D_\lambda \times F_\lambda + \sum_{\mu\neq\lambda} D_{\lambda\mu} \times F_\mu$$
$$- \sum_{\mu\neq\lambda} m_\mu \mathcal{L}_{\lambda\mu} \times \ddot{D}_{\lambda\mu} + m \sum_{\mu\neq\lambda} D_{\lambda\mu} \times \ddot{D}_{\mu\lambda} \tag{3}$$

where

$$D_\lambda = - \sum_{\mu\neq\lambda} m_\mu m^{-1} \mathcal{L}_{\lambda\mu} \tag{4}$$

and

$$D_{\lambda\mu} = D_\lambda + \mathcal{L}_{\lambda\mu} \tag{5}$$

From Eq. (3) it can be shown that

$$- \sum_{\mu\neq\lambda} m_\mu \mathcal{L}_{\lambda\mu} \times \ddot{D}_{\lambda\mu} = -X_\lambda \cdot \dot{\omega}_\lambda - \omega_\lambda \times X_\lambda \cdot \omega_\lambda \tag{6}$$

where $X_\lambda$ is the dyadic defined in Ref. 1 as

$$X_\lambda = \left(m_\lambda D^2_\lambda + \sum_{\mu\neq\lambda} m_\mu D^2_{\lambda\mu}\right) 1 - \left(m_\lambda D_\lambda D_\lambda + \sum_{\mu\neq\lambda} m_\lambda D_{\lambda\mu} D_{\lambda\mu}\right) \tag{7}$$

Equation (6) may be used to determine the equation of motion of the body under the influence of the gravity gradient torque presented as Eq. (19) in Ref. 1 and similarly as Eq. (1) in Ref. 2.

Although these equations were derived under the assumption that relative motion about a joint was purely rotational, they are readily modified to account for joints which may move with respect to the bodies they connect. In writing Eq. (6), we have assumed that the vector $D_{\lambda\mu}$ is fixed in body $\lambda$ so that

$$\ddot{D}_{\lambda\mu} = \dot{\omega}_\lambda \times D_{\lambda\mu} + \omega_\lambda \times (\omega_\lambda \times D_{\lambda\mu}) \tag{8}$$

If, however, motion of a joint connecting body $\lambda$ with respect to body $\lambda$ (e.g. along the surface of body $\lambda$) is allowed, $D_{\lambda\mu}$ will no longer be fixed in body $\lambda$ and we obtain

$$\ddot{D}_{\lambda\mu} = \ddot{D}^R_{\lambda\mu} + 2\omega_\lambda \times \dot{D}^R_{\lambda\mu} + \dot{\omega}_\lambda \times D_{\lambda\mu} + \omega_\lambda \times (\omega_\lambda \times D_{\lambda\mu}) \tag{9}$$

where superscript $R$ indicates differentiation with respect to time relative to the reference frame fixed in body $\lambda$. Substitution of Eq. (9) in Eq. (6) yields

$$- \sum_{\mu\neq\lambda} m_\mu \mathcal{L}_{\lambda\mu} \times \ddot{D}_{\lambda\mu} = -X_\lambda \cdot \dot{\omega}_\lambda - \omega_\lambda \times X_\lambda \cdot \omega_\lambda$$
$$- \sum_{\mu\neq\lambda} m_\mu \mathcal{L}_{\lambda\mu} \times [\ddot{D}^R_{\lambda\mu} + 2\omega_\lambda \times \dot{D}^R_{\lambda\mu}] \tag{10}$$

The terms which now appear in Eq. (10) but are absent from the original Eq. (6) influence the final result in the following way. The attitude equation for body $\lambda$, including the effect of the gravity gradient torque, is given in Ref. 2 as

$$\sum_{\mu\epsilon S} \Phi_{\lambda\mu} \cdot \dot{\omega}_\mu = E_\lambda + \sum_{j\epsilon J_\lambda} T^C_{\lambda j} \tag{11}$$

where $\Phi_{\lambda\mu}$ is the dyadic

$$\Phi_{\lambda\lambda} = \Phi_\lambda + m_\lambda (D^2_\lambda 1 - D_\lambda D_\lambda) + \sum_{\mu\neq\lambda} m_\mu (D^2_{\lambda\mu} 1 - D_{\lambda\mu} D_{\lambda\mu}) \tag{12}$$

$$\phi_{\lambda\mu} (\mu\neq\lambda) = -m (D_{\mu\lambda} \cdot D_{\mu\lambda} 1 - D_{\mu\lambda} D_{\mu\lambda}) \tag{13}$$

and $E_\lambda$ is the vector

$$E_\lambda = 3\gamma\rho^{-3} \hat{\rho} \times \Phi_{\lambda\lambda} \cdot \hat{\rho} - \omega_\lambda \times \Phi_{\lambda\lambda} \cdot \omega_\lambda + T'_\lambda$$
$$+ \sum_{j\epsilon J_\lambda} T^{SD}_{\lambda j} + D_\lambda \times F'_\lambda + \sum_{\mu\neq\lambda} D_{\lambda\mu}$$
$$\times \{F'_\mu + m\omega_\mu \times (\omega_\mu \times D_{\mu\lambda}) + m\gamma\rho^{-3} (1 - 3\hat{\rho}\hat{\rho}) \cdot D_{\mu\lambda}\} \tag{14}$$

The changes necessitated in Eq. (6), which appear explicitly in Eq. (10), yield a modified version of the attitude equation (11) for body $\lambda$,

$$\sum_{\mu\epsilon S} \Phi_{\lambda\mu} \cdot \dot{\omega}_\mu = E_\lambda + \sum_{j\epsilon J_\lambda} T^C_{\lambda j} - \sum_{\mu\neq\lambda} m_\mu \mathcal{L}_{\lambda\mu}$$
$$\times [\ddot{D}^R_{\lambda\mu} + 2\omega_\lambda \times \dot{D}^R_{\lambda\mu}] \tag{15}$$

Since we allow motion of the joint on body $\mu$ leading to body $\lambda$ relative to the mass center of body $\mu$, we conclude from Eq. (3) that we also need to use the more complete form of the second derivative of $D_{\mu\lambda}$, as in Eq. (9), rather than the form given by Eq. (8). This acceleration appears in $E_\lambda$ of Eq. (14) so that

$$m[\ddot{D}^R_{\mu\lambda} + 2\omega_\mu \times \dot{D}^R_{\mu\lambda} + \dot{\omega}_\mu \times (\omega_\mu \times D_{\mu\lambda})]$$

must be substituted for the term $m\omega_\mu \times (\omega_\mu \times D_{\mu\lambda})$ in $E_\lambda$, when motion of the joint relative to body $\mu$ is allowed.

Explicit elimination of the constraint torques, $T^C_{\lambda j}$, from Eq. (15) is accomplished as described in Ref. 2 by summing Eq. (15) over $S$ to yield three equations free of $T^C_{\lambda j}$. The additional $r$-3 equations necessary result from first summing Eq. (15) over all bodies to one side of a joint $j$ to isolate $T^C_{\lambda j}$ at this joint. Then, since $T^C_{\lambda j}$ is orthogonal to gimbal axes $\hat{g}_j$ at joint $j$, $r$-3 equations free of constraint torques are obtained by taking the scalar product of each $\hat{g}_j$ with the sum of Eq. (15). The following equations of motion, free of components of the constraint torques, result:

$$
\begin{bmatrix} a_{00} & \underline{a}_{01} & \underline{a}_{02}\cdots\underline{a}_{0,r-3} \\ \underline{a}_{10} & & \\ \cdot & & \\ \cdot & & \\ \underline{a}_{r-3,0} & & \end{bmatrix}
\begin{bmatrix} \dot{\underline{\omega}}_0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \\ \ddot{\gamma}_{r-3} \end{bmatrix} =
$$

$$
\begin{bmatrix}
\sum_\lambda \underline{E}^*_\lambda - \sum_\lambda \sum_{\mu\neq\lambda} m_\mu \underline{\mathcal{L}}_{\lambda\mu} \times [\ddot{\underline{D}}^R_{\lambda\mu} + 2\underline{\omega}_\lambda \times \dot{\underline{D}}^R_{\lambda\mu}] \\
\quad + \sum_\mu \sum_{\mu\neq\lambda} \underline{D}_{\lambda\mu} \times m[\ddot{\underline{D}}^R_{\mu\lambda} + 2\underline{\omega}_\mu \times \dot{\underline{D}}^R_{\mu\lambda}] \\
\hat{\underline{g}}_1 \cdot \Big\{ \sum_\lambda \epsilon_{1\lambda} \underline{E}^*_\lambda - \sum_\lambda \epsilon_{1\lambda} \sum_{\mu\neq\lambda} m_\mu \underline{\mathcal{L}}_{\lambda\mu} \times [\ddot{\underline{D}}^R_{\lambda\mu} + 2\underline{\omega}_\lambda \times \dot{\underline{D}}^R_{\lambda\mu}] \\
\quad + \sum_\lambda \epsilon_{1\lambda} \sum_{\mu\neq\lambda} \underline{D}_{\lambda\mu} \times m[\ddot{\underline{D}}^R_{\mu\lambda} + 2\underline{\omega}_\mu \times \dot{\underline{D}}^R_{\mu\lambda}] \Big\} \\
\vdots \\
\hat{\underline{g}}_{r-3} \cdot \Big\{ \sum_\lambda \epsilon_{r-3,\lambda} \underline{E}^*_\lambda - \sum_\lambda \epsilon_{r-3,\lambda} \sum_{\mu\neq\lambda} m_\mu \underline{\mathcal{L}}_{\lambda\mu} \times [\ddot{\underline{D}}^R_{\lambda\mu} + 2\underline{\omega}_\lambda \\
\quad \times \dot{\underline{D}}^R_{\lambda\mu}] + \sum_\lambda \epsilon_{r-3,\lambda} \sum_{\mu\neq\lambda} \underline{D}_{\lambda\mu} \times m[\ddot{\underline{D}}^R_{\mu\lambda} + 2\underline{\omega}_\mu \times \dot{\underline{D}}^R_{\mu\lambda}] \Big\}
\end{bmatrix}
$$

where

$$a_{00} = \sum_\lambda \sum_\mu \Phi_{\lambda\mu}, \text{ a dyadic}$$

$$a_{0k} = \sum_\lambda \sum_\mu \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot \hat{\underline{g}}_k, \text{ a vector}$$

$$\underline{a}_{j0} = \hat{\underline{g}}_j \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda} \Phi_{\lambda\mu}, \text{ a vector}$$

$$a_{ik} = \hat{\underline{g}}_i \cdot \sum_\lambda \sum_\mu \epsilon_{i\lambda} \epsilon_{k\mu} \Phi_{\lambda\mu} \cdot \hat{\underline{g}}_k, \text{ a scalar}$$

$$\underline{E}^*_\lambda = \underline{E}_\lambda - \sum_\mu \Phi_{\lambda\mu} \cdot \sum_k \epsilon_{k\mu} \dot{\gamma}_k \dot{\hat{\underline{g}}}_k$$

and

$\epsilon_{i\mu} = 1$, if $\hat{\underline{g}}_i$ belongs to a joint anywhere on the chain of bodies connecting body $\mu$ and the reference body
$= 0$, otherwise (e.g., if $\mu = 0$)

The use and numerical integration of Eqs. (16) will be little changed from that of the original equations except that the joint velocities and accelerations with respect to the bodies they connect, $\dot{\underline{\mathcal{L}}}^R_{\lambda\mu}$ and $\ddot{\underline{\mathcal{L}}}^R{}_{\lambda\mu}$, respectively, must be prescribed and hence $\dot{\underline{D}}^R_{\lambda\mu}$ and $\ddot{\underline{D}}^R_{\lambda\mu}$ determined from Eq. (5). To determine the spacecraft attitude with respect to an external frame, a set of first-order equations, such as Euler angle rate equations, must be added. We have found that as a check on the accuracy of the computer code, which can be lengthy, it is helpful to add a routine which determines total system angular momentum about the mass center and check for its conservation when no external torques are considered.

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# Analytic Fourier Transform for a Class of Finite-Time Control Problems


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## Introduction

A COMMON method for investigating the effectiveness of various control designs consists of studying frequency domain characteristics of the control, by numerically evaluating the required Fourier transform. For finite-time open- and closed-loop control problems, this can be accomplished by either numerically integrating the integral definition of Fourier transform for each frequency of interest, or using a fast Fourier transform algorithm. Alternatively, we present in this Note a computationally efficient closed-form solution for the Fourier transform of finite-time open- and closed-loop control problems, where the dynamics of the control is governed by matrix exponentials.

## Problem Formulation

The fundamental definition of the complex Fourier transform follows as

$$\bar{u}(\omega) = \int_0^T u(t) e^{-i\omega t} \mathrm{d}t \qquad (n \times 1) \tag{1}$$

where $u(t)$ is assumed to be given by[1-4]

$$u(t) = A e^{Bt} b \qquad (n \times 1) \tag{2}$$

where $A$ is $n \times m$, $B$ is $m \times m$, $e^{(\cdot)}$ is the matrix exponential, and $b$ is $m \times 1$.

Introducing Eq. (2) into Eq. (1) yields

$$\bar{u}(\omega) = A \xi(\omega) \tag{3}$$

where

$$\xi(\omega) = \int_0^T e^{Bt} b e^{-i\omega t} \mathrm{d}t \tag{4}$$


Submitted March 14, 1984; revision received July 30, 1984.



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